Example 11 Find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum.

Solution From Fig 8.20, the vertex of the parabola $y^2 = 4ax$ is at origin (0, 0). The equation of the latus rectum LSL′ is $x = a$. Also, parabola is symmetrical about the x-axis.

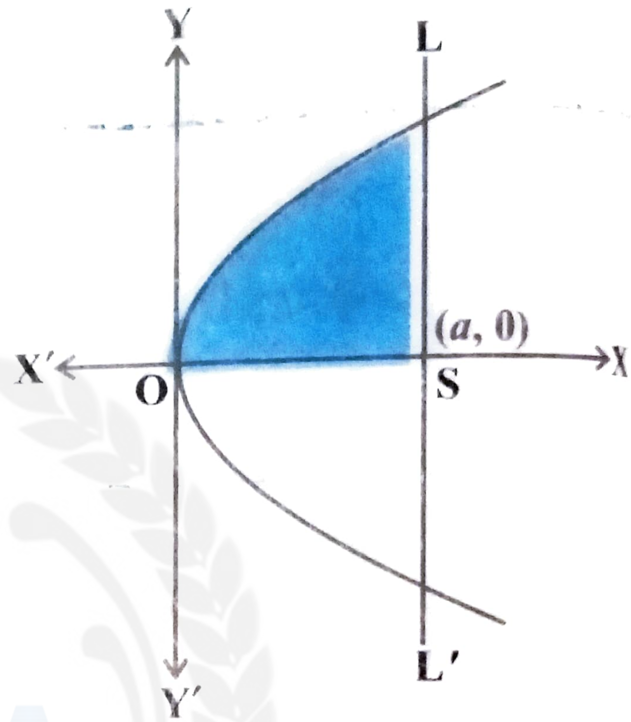

Fig 8.20

The required area of the region OLL′O

$$= 2\,(\text{area of the region OLSO})$$

$$= 2\int_0^a y\,dx = 2\int_0^a \sqrt{4ax}\,dx$$

$$= 2\times 2\sqrt{a}\int_0^a \sqrt{x}\,dx$$

$$= 4\sqrt{a}\times\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^a$$

$$= \frac{8}{3}\sqrt{a}\left[a^{\frac{3}{2}}\right] = \frac{8}{3}a^2$$

Example 12 Find the area of the region bounded by the line $y = 3x + 2$, the x-axis and the ordinates $x = -1$ and $x = 1$.

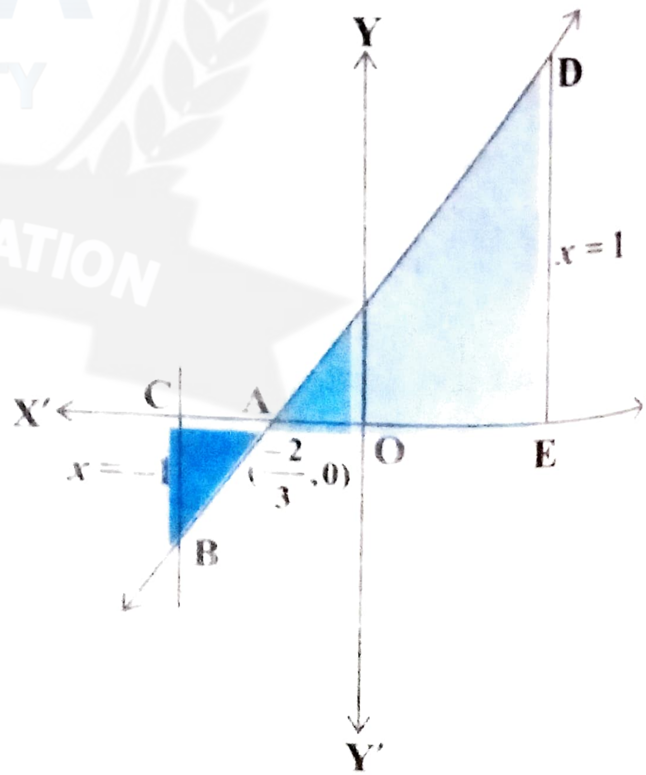

Fig 8.21

Solution As shown in the Fig 8.21, the line $y = 3x + 2$ meets x-axis at $x = \frac{-2}{3}$ and its graph lies below x-axis for $x \in \left(-1, \frac{-2}{3}\right)$ and above x-axis for $x \in \left(\frac{-2}{3}, 1\right)$.

The required area = Area of the region ACBA + Area of the region ADEA

$$= \left|\int_{-1}^{\frac{-2}{3}} (3x+2)dx\right| + \int_{\frac{-2}{3}}^{1} (3x+2)dx$$

$$= \left|\left[\frac{3x^2}{2}+2x\right]_{-1}^{\frac{-2}{3}}\right| + \left[\frac{3x^2}{2}+2x\right]_{\frac{-2}{3}}^{1} = \frac{1}{6}+\frac{25}{6}=\frac{13}{3}$$

Example 13 Find the area bounded by the curve $y = \cos x$ between $x = 0$ and $x = 2\pi$.

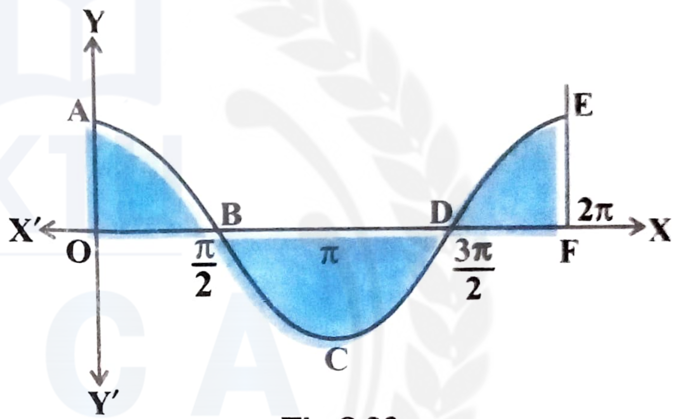

Fig 8.22

Solution From the Fig 8.22, the required area = area of the region OABO + area of the region BCDB + area of the region DEFD.

Thus, we have the required area

$$= \int_0^{\frac{\pi}{2}} \cos x\,dx + \left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx\right| + \int_{\frac{3\pi}{2}}^{2\pi} \cos x\,dx$$

$$= [\sin x]_0^{\frac{\pi}{2}} + \left|[\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right| + [\sin x]_{\frac{3\pi}{2}}^{2\pi}$$

$$= 1 + 2 + 1 = 4$$

Example 13 Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three equal parts.

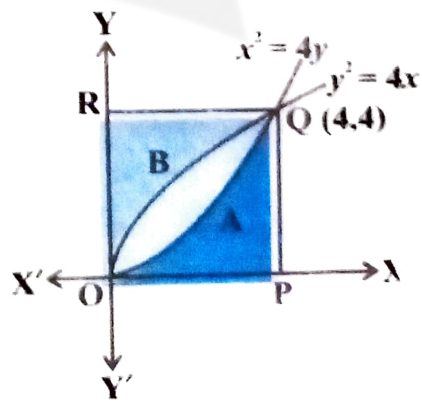

Fig 8.23

Solution Note that the point of intersection of the parabolas $y^2 = 4x$ and $x^2 = 4y$ are (0, 0) and (4, 4) as shown in the Fig 8.23.

Now, the area of the region OAQBO bounded by curves $y^2 = 4x$ and $x^2 = 4y$.

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \left[2 \times \frac{2}{3} x^{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \quad \text{... (1)}$$

Again, the area of the region OPQAO bounded by the curves $x^2 = 4y$, $x = 0$, $x = 4$ and x-axis

$$= \int_0^4 \frac{x^2}{4} dx = \frac{1}{12}\left[x^3\right]_0^4 = \frac{16}{3} \quad \text{... (2)}$$

Similarly, the area of the region OBQRO bounded by the curve $y^2 = 4x$, y-axis, $y = 0$ and $y = 4$

$$= \int_0^4 x dy = \int_0^4 \frac{y^2}{4} dy = \frac{1}{12}\left[y^3\right]_0^4 = \frac{16}{3} \quad \text{... (3)}$$

From (1), (2) and (3), it is concluded that the area of the region OAQBO = area of the region OPQAO = area of the region OBQRO, i.e., area bounded by parabolas $y^2 = 4x$ and $x^2 = 4y$ divides the area of the square in three equal parts.

Example 14 Find the area of the region

$$\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$$

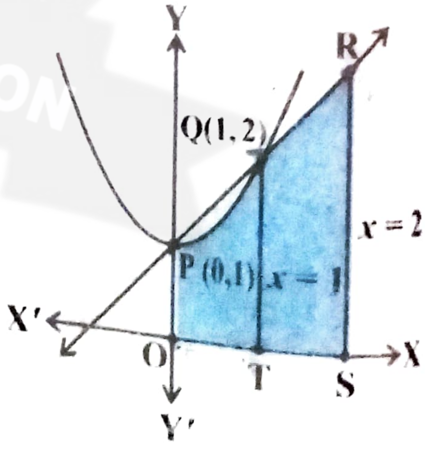

Fig 8.24

Solution Let us first sketch the region whose area is to be found out. This region is the intersection of the following regions.

$$A_1 = \{(x, y) : 0 \le y \le x^2 + 1\},$$

$$A_2 = \{(x, y) : 0 \le y \le x + 1\}$$

and

$$A_3 = \{(x, y) : 0 \le x \le 2\}$$

The points of intersection of $y = x^2 + 1$ and $y = x + 1$ are points P(0, 1) and Q(1, 2). From the Fig 8.24, the required region is the shaded region OPQRSTO whose area

= area of the region OTQPO + area of the region TSRQT

$$= \int_0^1 (x^2 + 1)\, dx + \int_1^2 (x + 1)\, dx \qquad \text{(Why?)}$$

$$= \left[\left(\frac{x^3}{3} + x\right)\right]_0^1 + \left[\left(\frac{x^2}{2} + x\right)\right]_1^2$$

$$= \left[\left(\frac{1}{3} + 1\right) - 0\right] + \left[(2+2) - \left(\frac{1}{2} + 1\right)\right] = \frac{23}{6}$$

Summary

- The area of the region bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ $(b > a)$ is given by the formula: Area $= \int_a^b y\,dx = \int_a^b f(x)\,dx$.
- The area of the region bounded by the curve $x = \phi(y)$, y-axis and the lines $y = c$, $y = d$ is given by the formula: Area $= \int_c^d x\,dy = \int_c^d \phi(y)\,dy$.

- The area of the region enclosed between two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ is given by the formula,

$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx \text{, where, } f(x) \geq g(x) \text{ in } [a, b]$$

- If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, $a < c < b$, then

$$\text{Area} = \int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx.$$